Week 7 Worksheet Solutions

TA: Emil Geisler and Caleb Partin

May 16, 2024

Exercise 1. Determine whether the following sets of vectors are *orthonormal* (orthogonal and unit length):

(a) $\begin{bmatrix}3/5\\4/5\end{bmatrix}, \begin{bmatrix}-4/5\\3/5\end{bmatrix}$.

(b) $\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$.

(c) $\begin{bmatrix}2/3\\-1/3\\2/3\end{bmatrix}, \begin{bmatrix}-1/3\\2/3\\2/3\end{bmatrix}, \begin{bmatrix}2/3\\2/3\\-1/3\end{bmatrix}$

(d) $\begin{bmatrix}a\\a\end{bmatrix}, \begin{bmatrix}a\\-b\end{bmatrix}, \begin{bmatrix}b\\a\end{bmatrix}$ for $a, b \in \mathbb{R}$.

(a) Dot product is zero, and $(3/5)^2 + (4/5)^2 = 1$, so yes.

(b) No, since $|\begin{bmatrix}1\\-1\end{bmatrix}| = 2 \neq 1$.

(c) Yes, dot product between any pair is zero, and $|\begin{bmatrix}2/3\\2/3\\-1/3\end{bmatrix}| = 4/9+4/9+1/9 = 1$ (and similarly for the other two).

(d) These cannot be orthonormal since otherwise they would form an orthornomal basis for $\mathbb{R}^2$, which is dimension 2 (and thus cannot have a basis of size 3). Alternatively for the first and last vector to be orthogonal, we must have $a = -b$, but for the first and second vector to be orthogonal we must have $a = b$, so $b = -b$ and thus $b = 0$, and so for $\begin{bmatrix}a\\a\end{bmatrix}$ and $\begin{bmatrix}a\\0\end{bmatrix}$ to be orthogonal we have to have $a = 0$ so they aren't unit length.

Exercise 2. Find a basis for $W^\perp$, where

$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \right\}$$

(Hint: How can we relate $W^\perp$ to subspaces where we know how to find a basis?)

A vector v is in $W^\perp$ if and only if $v \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = 0$ and $v \cdot \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = 0$ (to check if a vector is orthogonal to a subspace, we only need to check that it is orthogonal to the basis vectors). However, notice that if we let A be the matrix with the basis vectors as rows:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

that this is the same as asking that v is in $\ker A$. Thus, we need to find a basis for kernel of A:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

So letting $x_3 = t$, $x_4 = s$ for $t, s \in \mathbb{R}$, we see that $x_1 = x_3 + 2x_4 = t + 2s$ and $x_2 = -2x_3 - 3x_4 = -2t - 3s$, giving a general solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t + 2s \\ -2t - 3s \\ t \\ s \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

And so we can see:

$$W^\perp = \ker(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Exercise 3. Find the orthogonal projection of $\begin{bmatrix}5\\5\\5\end{bmatrix}$ onto the subspace

$$V = \text{span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}-2\\1\\1\end{bmatrix}\right\}$$

Let's first normalize the basis vectors for V so we get a basis of unit vectors:

$$\left\|\begin{bmatrix}1\\2\\1\end{bmatrix}\right\| = \sqrt{1+2^2+1} = \sqrt{6}$$

$$\left\|\begin{bmatrix}-2\\1\\1\end{bmatrix}\right\| = \sqrt{(-2)^2+1+1} = \sqrt{6}$$

$$V = \text{span}\left\{\frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\1\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}-2\\1\\1\end{bmatrix}\right\}$$

Then the projection formula tells us:

$$\begin{aligned} proj_V\left(\begin{bmatrix}5\\5\\5\end{bmatrix}\right) &= \left(\begin{bmatrix}5\\5\\5\end{bmatrix} \cdot \frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\1\end{bmatrix}\right)\frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\1\end{bmatrix} + \left(\begin{bmatrix}5\\5\\5\end{bmatrix} \cdot \frac{1}{\sqrt{6}}\begin{bmatrix}-2\\1\\1\end{bmatrix}\right)\frac{1}{\sqrt{6}}\begin{bmatrix}-2\\1\\1\end{bmatrix} \\ &= \left(\frac{20}{\sqrt{6}}\right)\frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\1\end{bmatrix} + (0)\frac{1}{\sqrt{6}}\begin{bmatrix}-2\\1\\1\end{bmatrix} = \frac{5}{9}\begin{bmatrix}1\\2\\1\end{bmatrix} \end{aligned}$$

Exercise 4. For each of the following vectors $\vec{v}$, find the decomposition $v^{||} + v^{\perp}$ with respect to the subspace

$$V = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(a) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

First we normalize all of our basis vectors to get

$$V = \text{span}\left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\}$$

(a)

$$v^{||} = \text{proj}_V\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

$$v^{\perp} = v - v^{||} = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \end{bmatrix}$$

(b)

$$v^{||} = \text{proj}_V\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right) = 2 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + 0 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} + 0 \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v^{\perp} = v - v^{||} = \vec{0}$$

(c)

$$v^{||} = \text{proj}_V\left(\begin{bmatrix}0\\0\\0\\0\end{bmatrix}\right) = 0\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{bmatrix} + 0\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\end{bmatrix} + 0\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$v^{\perp} = v - v^{||} = \vec{0}$$

Exercise 5. Let $V = \text{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$ be a subspace of $\mathbb{R}^n$ where the vectors $\vec{v}_1, \ldots, \vec{v}_k$ give an orthonormal basis for V.

(a) If $\vec{w} \in V$, show that $\text{proj}_V(\vec{w}) = \vec{w}$.

(b) If $\vec{w} \in V^\perp$, show that $\text{proj}_{V^\perp}(\vec{w}) = 0$.

(a) Since $\vec{w} \in V$, we know that $\vec{w} = c_1\vec{v_1} + \cdots + c_k\vec{v_k}$ for some scalars $c_1, \ldots, c_k$. Moreover, note that since $\vec{v_1}, \ldots, \vec{v_k}$ form an orthonormal basis, we know that $\vec{v_i} \cdot \vec{v_j} = 0$ for $i \neq j$ and $\vec{v_i} \cdot \vec{v_j} = 1$ when $i = j$. Using this and the fact that the dot product distributes over sums, we can compute the projection:

$$\begin{aligned}
\text{proj}_V(\vec{w}) &= (\vec{w} \cdot \vec{v_1})\vec{v_1} + \cdots + (\vec{w} \cdot \vec{v_k})\vec{v_k} = ((c_1\vec{v_1} + \cdots + c_k\vec{v_k}) \cdot \vec{v_1})\vec{v_1} + \cdots + ((c_1\vec{v_1} + \cdots + c_k\vec{v_k}) \cdot \vec{v_k})\vec{v_k} \\
&= (c_1\vec{v_1} \cdot \vec{v_1} + \cdots + c_k\vec{v_k} \cdot \vec{v_1})\vec{v_1} + \cdots + ((c_1\vec{v_1} \cdot \vec{v_k} + \cdots + c_k\vec{v_k} \cdot \vec{v_k})\vec{v_k} \\
&= c_1\vec{v_1} + \cdots + c_k\vec{v_k} = \vec{w}
\end{aligned}$$

(b) If $\vec{w} \in V^\perp$, then $\vec{w}$ must be orthogonal to every basis vector for W:

$$\text{proj}_{V^\perp}(\vec{w}) = (\vec{w} \cdot \vec{v_1})\vec{v_1} + \cdots + (\vec{w} \cdot \vec{v_k})\vec{v_k} = 0\vec{v_1} + \cdots + 0\vec{v_k} = \vec{0}$$